On A Fixed-Charge Grid Network Polyhedron

Minjiao Zhang[1] Simge Küçükyavuz[1] Hande Yaman[2]
Supported, in part, by NSF #0917952.

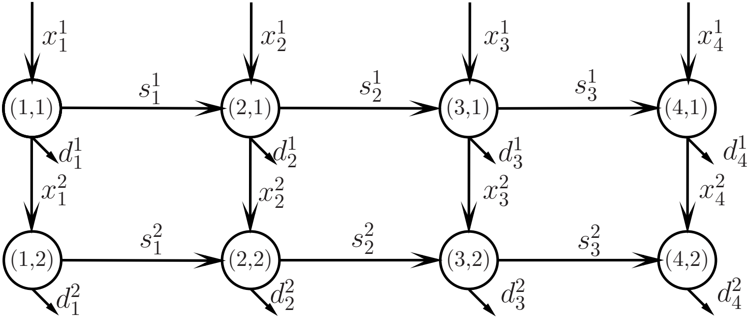
[1]Integrated Systems Engineering, The Ohio State University, USA
[2]Department of Industrial Engineering, Bilkent University, Turkey

Integer Programming Down Under Workshop, 2011

Outline

Introduction

Fixed-Charge Grid Network



Motivation

- Distribution or production systems with multiple echelons. (e.g., the regional distribution centers and local retailers)
- Two-echelon lot-sizing problem in series and with intermediate demands (2-ULS).
- Fixed and variable order costs, and variable holding costs at each echelon.
- Goal: Determine the order plan over a finite horizon to meet the demand at both echelons in each period with the minimum total cost:

Related work

- Single echelon lot-sizing:
  Wagner and Whitin, 1958; Krarup and Bilde, 1977; Eppen and Martin, 1987; Barany et al, 1984;
- Serial multi-echelon lot-sizing without intermediate demands:
  Veinott, 1969; Zangwill, 1969; Pochet and Wolsey, 2006; Melo and Wolsey, 2010
- Complex bill-of-materials:
  Gaglioppa, Miller and Benjaafar, 2008; Akartunalı and Miller, 2009; Wu et al., 2011.

Notation

Parameters

- $d_t^i \geq 0$: the demand in period t at the ith echelon, and $d_{tk}^i = \sum_{j=t}^{k} d_j^i$.
- f_t^i: fixed cost incurred in period t at echelon i.
- $\tilde{c}_t^i$: variable cost incurred in period t at echelon i.
- h_t^i: holding cost at echelon i at the end of period t.
- $[i, j]$: the interval $\{i, i+1, \ldots, j\}$ for $i \leq j$, and $[i, j] = \emptyset$ for $i > j$.

Variables

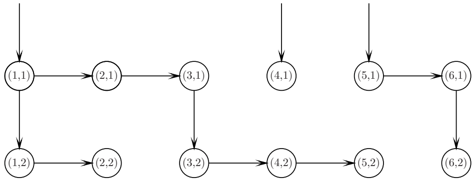
- x_t^i: the order quantity at echelon i in period t.
- s_t^i: the inventory at echelon i at the end of period t.
- y_t^i: the order setup variable at echelon i in period t; $y_t^i = 1$ if $x_t^i > 0$; $y_t^i = 0$ otherwise.

Formulation

$$
\begin{aligned}
\min \sum_{i=1}^{2} \sum_{t=1}^{n} (f_t^i y_t^i + \tilde{c}_t^i x_t^i + h_t^i s_t^i) & & (1)\\
\text{s.t. } s_{t-1}^1 + x_t^1 = d_t^1 + x_t^2 + s_t^1 & \quad t \in [1, n], & (2)\\
s_{t-1}^2 + x_t^2 = d_t^2 + s_t^2 & \quad t \in [1, n], & (3)\\
s_0^i = s_n^i = 0 & \quad i \in [1, 2], & (4)\\
x_t^1 \leq (d_{tn}^1 + d_{tn}^2) y_t^1 & \quad t \in [1, n], & (5)\\
x_t^2 \leq d_{tn}^2 y_t^2 & \quad t \in [1, n], & (6)\\
y_t^i \in \{0, 1\} & \quad t \in [1, n], i \in [1, 2], & (7)\\
x_t^i \geq 0 & \quad t \in [1, n], i \in [1, 2], & (8)\\
s_t^i \geq 0 & \quad t \in [1, n], i \in [1, 2]. & (9)
\end{aligned}
$$

Extreme Point Solutions

- (i_1, i_2, j_1, j_2) is defined as a **regeneration interval** if for $i_1 \leq j_1 \leq j_2$, $s^1_{i_1-1} = s^1_{i_2} = s^2_{j_1-1} = s^2_{j_2} = 0$, $x^1_{i_1} = d^1_{i_1 i_2} + d^2_{j_1 j_2}$, or for $j_1 = j_2 + 1$, $s^1_{i_1-1} = s^1_{i_2} = 0$, $x^1_{i_1} = d^1_{i_1 i_2}$.
- (j_1, j_2) is defined as a **regeneration subinterval** with $1 \leq j_1 \leq j_2 \leq n$ and $s^2_{j_1-1} = s^2_{j_2} = 0$, $x^2_{j_1} = d^2_{j_1 j_2}$.



$(1, 3, 1, 5)$, $(4, 4, 5, 4)$, $(5, 6, 6, 6)$ are regeneration intervals. $(3, 5)$, $(1, 2)$, $(6, 6)$ are regeneration subintervals.

Dynamic Programming Recursion

- $G(i_2, j_2)$: the minimum cost of satisfying the demand in periods 1 to i_2 at the first echelon and the demand in periods 1 to j_2 at the second echelon, $0 \leq i_2 \leq j_2 \leq n$. $G(0,0) = 0$ and $G(0,k) = \infty$ if $k \in [1,n]$.
- $H(j_1, j_2)$: the minimum cost to satisfy the demand in periods j_1 to j_2 at the second echelon, $1 \leq j_1 \leq n+1, 0 \leq j_2 \leq n$. $H(j_1, j_2) = 0$ if $j_1 > j_2$.

Forward DP recursion

For $1 \leq i_2 \leq j_2 \leq n$,

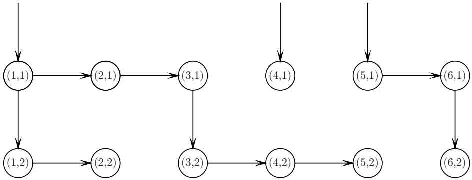
$$G(i_2, j_2) = \min_{\substack{1 \leq i_1 \leq i_2 \\ i_1 \leq j_1 \leq j_2+1}} \{G(i_1 - 1, j_1 - 1) + f^1_{i_1} + c^1_{i_1} d^1_{i_1 i_2} + c^1_{i_1} d^2_{j_1 j_2} + H(j_1, j_2)\};$$

for $1 \leq j_1 \leq j_2 \leq n$, $H(j_1, j_2) = \min_{j_1 \leq j_3 \leq j_2} \{H(j_1, j_3 - 1) + f^2_{j_3} + c^2_{j_3} d^2_{j_3 j_2}\}$.

- Running time of DP: $O(n^4)$.
- If $d^1_j = 0, \forall j$, the DP is equivalent to that of Melo and Wolsey (2010).

DP-based extended formulation (DPEF)

- DPEF is the dual of the LP model based on the DP recursions.
- Variables:
  - $v_{i_1 i_2 j_1 j_2} = 1$ if (i_1, i_2, j_1, j_2) is a regeneration interval; otherwise, $v_{i_1 i_2 j_1 j_2} = 0$.
  - $w_{j_1 j_3 j_2} = 1$ if (j_3, j_2) is a regeneration subinterval; otherwise, $w_{j_1 j_3 j_2} = 0$.



$v_{1315} = 1$, $v_{4454} = 1$, $v_{5666} = 1$ and $w_{135} = 1$, $w_{112} = 1$, $w_{666} = 1$.

- DPEF is **tight** and **compact**: $O(n^4)$ variables and $O(n^3)$ constraints.

Valid Inequalities

Define $\beta(T, k)$ as the set of consecutive elements in set T starting from k, where if $k \notin T, \beta(T, k) = \emptyset$.

Theorem (*Zhang, K., Yaman, 2011*)

For $0 \leq k \leq l \leq n$, let $T_1 \subseteq [1, k]$, $C \subseteq [1, k]$, $T_2 = C \cup [k+1, l]$ and $T_3 \subseteq T_2$. Then the 2-echelon inequality

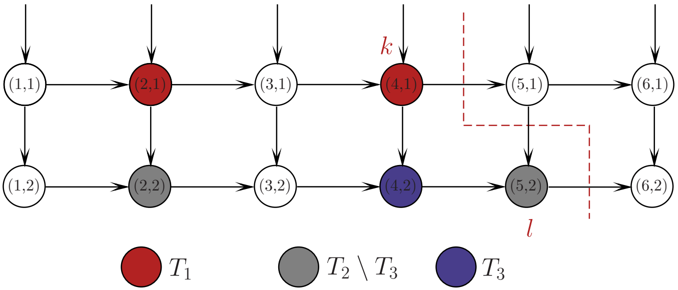
$$\sum_{j \in [1,k] \setminus T_1} x_j^1 + \sum_{j \in T_1} \phi_j y_j^1 + \sum_{j \in T_2 \setminus T_3} x_j^2 + \sum_{j \in T_3} \psi_j y_j^2 \geq d_{1k}^1 + d_{1l}^2$$

is valid for 2-ULS and facet-defining under certain conditions, where $\psi_j = \sum_{i \in \beta(T_2, j)} d_i^2$ and $\phi_j = d_{jk}^1 + d_{jl}^2 - \psi_j$.

2-echelon inequalities subsume (ℓ, S) inequalities of Barany et al. (1984)

Valid Inequalities

e.g. For $n = 6$, $k = 4$, $l = 5$, $d_j^i = 1 \quad \forall i = 1, 2,\ j = 1, \ldots, 6$,
$x_1^1 + 6y_2^1 + x_3^1 + y_4^1 + x_2^2 + 2y_4^2 + x_5^2 \geq 9$ is a valid inequality with following choices of T_1, T_2 and T_3.



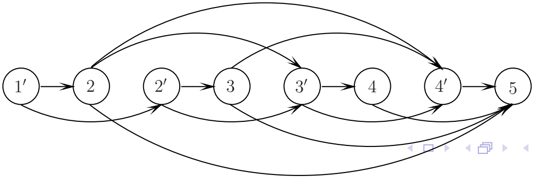
Valid Inequalities

Proposition 1 (*Zhang, K., Yaman, 2011*)

Given a fractional point $(\mathbf{x^1}, \mathbf{y^1}, \mathbf{x^2}, \mathbf{y^2}) \in \mathbb{R}^{4n}$, there is an $O(n^4)$ algorithm to find the most violated 2-echelon inequality, if any.

Separation:

- Shortest path problem on the network $G = (V, A)$, for a given k:
  $V = \{1', 2, 2', \ldots, k, k', k+1\}$ and $A = \{(i', i+1) : i = 1, \ldots, k\}$
  $\cup\{(i', (i+1)') : i = 1, \ldots k\}$
  $\cup\{(i, j') : i \in [2, k-2], j \in [i+1, k-1]\}$.
- Node i represents $i \in T_2$, i' represents $i' \notin T_2$.
- For $k = 4$, the shortest path network is:



Facility Location Formulation for 2-ULS

- Variables:
  - z_{ut}^{11}: the order quantity in period u at the first echelon to satisfy the intermediate demand in period t, $1 \leq u \leq t \leq n$;
  - z_{ut}^{12}: the order quantity in period u at the first echelon to satisfy the demand at the second echelon in period t, $1 \leq u \leq t \leq n$;
  - z_{ut}^{22}: the order quantity in period u at the second echelon to satisfy the demand at the second echelon in period t, $1 \leq u \leq t \leq n$.

Facility Location Formulation for 2-ULS

$$\min \sum_{i=1}^{2} \sum_{t=1}^{n} (f_t^i y_t^i + c_t^i x_t^i)$$

$$\text{s.t.} \quad \sum_{u=1}^{t} z_{ut}^{11} = d_t^1 \qquad t \in [1,n], \tag{10}$$

$$\sum_{u=1}^{t} z_{ut}^{22} = d_t^2 \qquad t \in [1,n],$$

$$\sum_{u=1}^{j} z_{ut}^{12} \geq \sum_{u=1}^{j} z_{ut}^{22} \qquad t \in [1,n], j \in [1,t],$$

$$z_{ut}^{11} \leq d_t^1 y_u^1 \qquad t \in [1,n], u \in [1,t], \tag{11}$$

$$z_{ut}^{12} \leq d_t^2 y_u^1 \qquad t \in [1,n], u \in [1,t],$$

$$z_{ut}^{22} \leq d_t^2 y_u^2 \qquad t \in [1,n], u \in [1,t],$$

$$x_t^1 = \sum_{u=t}^{n} (z_{tu}^{11} + z_{tu}^{12}) \qquad t \in [1,n],$$

$$x_t^2 = \sum_{u=t}^{n} z_{tu}^{22} \qquad t \in [1,n],$$

$$z_{ut}^{11}, z_{ut}^{12}, z_{ut}^{22} \geq 0 \qquad t \in [1,n], u \in [1,t],$$

$$y_t^i \in \{0,1\} \qquad t \in [1,n], i \in [1,2].$$

Facility Location Formulation for 2-ULS

- UFL formulation $\equiv$ multi-commodity formulation (Rardin and Wolsey, 1993) after projecting out the inventory variables.
- Without intermediate demands, constraints (10) and (11) are redundant. (Pochet and Wolsey, 2006)

Proposition 2 (*Zhang, K., Yaman, 2011*)

2-echelon inequalities can be obtained by projecting the UFL formulation onto the $(\mathbf{x^1}, \mathbf{y^1}, \mathbf{x^2}, \mathbf{y^2})$ space.

Proof idea

Every valid projection inequality with 0-1 coefficients is a 2-echelon inequality.

Echelon Stock Formulation for 2-ULS

- Variables:
  - e_t^1: the total inventory at the first echelon at the end of period t, $t \in [1, n]$. Note that $e_t^1 = s_t^1 + s_t^2$.
  - e_t^2: the total inventory at the second echelon at the end of period t, $t \in [1, n]$. Note that $e_t^2 = s_t^2$.
- Echelon Stock Formulation:

$$
\begin{aligned}
\min & \sum_{i=1}^{2} \sum_{t=1}^{n} (f_t^i y_t^i + c_t^i x_t^i) & \\
\text{s.t. } & (5) - (8), & \\
& e_{t-1}^1 + x_t^1 = d_t^1 + d_t^2 + e_t^1 & t \in [1, n], \\
& e_{t-1}^2 + x_t^2 = d_t^2 + e_t^2 & t \in [1, n], \\
& e_0^i = e_n^i = 0 & i \in [1, 2], \\
& e_t^1 \geq e_t^2 & t \in [1, n], \\
& e_t^i \geq 0 & t \in [1, n], i \in [1, 2].
\end{aligned}
$$

Echelon Stock Formulation for 2-ULS

Proposition (*Zhang, K., Yaman, 2011*)

The natural formulation with 2-echelon inequalities is stronger than the echelon stock reformulation with (ℓ, S) inequalities.

Hierarchy of Formulations

From stronger to weaker:

- Projection of the DP-based extended formulation (DPEF);
- Projection of the UFL formulation (or equivalently multi-commodity formulation);
- Natural formulation with 2-echelon inequalities;
- Echelon stock formulation with (ℓ, S) inequalities.
- Natural formulation.

Computational Results

- Tests on **multi-item** 2-ULS with a mode constraint.
- Environment: 1 GHz Dual-Core AMD Opteron(tm) Processor 1218 with 2GB RAM.
- Solver: IBM ILOG CPLEX 12.0 with an hour time limit.
- Data generation:
  - Horizon: $n \in \{60, 90, 120\}$,
  - Ratios of fixed and unit order costs: $\beta \in \{500, 1000, 2500\}$,
  - Numbers of items $r \in \{5, 10\}$,
  - Capacities (the maximum number of items ordered in each period) $\kappa \in \{2, 3, 5\}$.
- Average of five instances for each combination of n, β, r, k.

Computational Results

	MIP				UFL			
$n.\beta.r.\kappa$	IGap	EGap	Time	Nodes	IGap	EGap	Time	Nodes
60.500.5.2	42.86	14.62	$\geq$3600	396140	0	0	1.75	0
60.500.10.3	42.29	20.80	$\geq$3600	302236	0	0	5.16	0
60.1000.5.3	40.11	17.13	$\geq$3600	584511	0	0	3.58	0
60.2500.10.5	28.03	2.54	$\geq$3600	914349	0	0	12.70	0
90.500.5.3	51.27	24.63	$\geq$3600	351115	0	0	9.33	0
90.1000.5.3	49.39	31.43	$\geq$3600	449264	0	0	10.42	0
90.1000.10.5	48.63	35.15	$\geq$3600	230387	0	0	17.31	0
90.2500.10.3	41.05	33.16	$\geq$3600	371711	0	0	51.17	0
120.500.10.3	55.15	30.54	$\geq$3600	115899	0	0	25.01	0
120.1000.5.3	55.30	38.40	$\geq$3600	304437	0	0	27.74	0
120.2500.5.2	49.03	42.49	$\geq$3600	385625	0	0	29.67	0
120.2500.10.5	48.98	46.01	$\geq$3600	193028	0	0	70.95	0

Concluding Remarks

- Size matters. (DPEF is too large.)
- UFL formulation is not tight in general, but extremely strong in practice.
- Complete linear description of 2-ULS in the original space is open (even with zero intermediate demands)
- Complete characterization of the UFL projection inequalities is open.
- Automatic reformulation ideas (van Roy and Wolsey, 1987)

Reference

- M. Zhang, S. Küçükyavuz, H. Yaman. 2011. A Polyhedral Study of Multi-Echelon Lot Sizing with Intermediate Demands, available at Optimization Online, May 2011.

Thanks!