On A Fixed-Charge Grid Network Polyhedron

Minjiao Zhang\textsuperscript{1}  Simge K"uc"ukyavuz\textsuperscript{1}  Hande Yaman\textsuperscript{2}
Supported, in part, by NSF \#0917952.

\textsuperscript{1}Integrated Systems Engineering, The Ohio State University, USA
\textsuperscript{2}Department of Industrial Engineering, Bilkent University, Turkey

Integer Programming Down Under Workshop, 2011
Outline

1. Introduction
2. Dynamic Program
3. Alternative Formulations
4. Computational Results
Fixed-Charge Grid Network
Motivation

- Distribution or production systems with multiple echelons. (e.g., the regional distribution centers and local retailers)
- Two-echelon lot-sizing problem in series and with intermediate demands (2-ULS).
- Fixed and variable order costs, and variable holding costs at each echelon.
- Goal: Determine the order plan over a finite horizon to meet the demand at both echelons in each period with the minimum total cost:
Related work

- **Single echelon lot-sizing:**
  Wagner and Whitin, 1958; Krarup and Bilde, 1977; Eppen and Martin, 1987; Barany et al, 1984;

- **Serial multi-echelon lot-sizing without intermediate demands:**
  Veinott, 1969; Zangwill, 1969; Pochet and Wolsey, 2006; Melo and Wolsey, 2010

- **Complex bill-of-materials:**
  Gaglioppa, Miller and Benjaafar, 2008; Akartunalı and Miller, 2009; Wu et al., 2011.
Notation

Parameters

- $d^i_t \geq 0$: the demand in period $t$ at the $i$th echelon, and $d^i_{tk} = \sum_{j=t}^k d^i_j$.
- $f^i_t$: fixed cost incurred in period $t$ at echelon $i$.
- $\tilde{c}^i_t$: variable cost incurred in period $t$ at echelon $i$.
- $h^i_t$: holding cost at echelon $i$ at the end of period $t$.
- $[i, j]$: the interval $\{i, i + 1, \ldots, j\}$ for $i \leq j$, and $[i, j] = \emptyset$ for $i > j$.

Variables

- $x^i_t$: the order quantity at echelon $i$ in period $t$.
- $s^i_t$: the inventory at echelon $i$ at the end of period $t$.
- $y^i_t$: the order setup variable at echelon $i$ in period $t$; $y^i_t = 1$ if $x^i_t > 0$; $y^i_t = 0$ otherwise.
Formulation

\[
\begin{align*}
\text{min} & \sum_{i=1}^{2} \sum_{t=1}^{n} (f_{i}^{i} y_{i}^{t} + \bar{c}_{i}^{i} x_{i}^{t} + h_{i}^{i} s_{i}^{t}) \\
\text{s.t.} & \quad s_{i}^{t-1} + x_{i}^{t} = d_{i}^{1} + x_{i}^{t} + s_{i}^{t} \quad t \in [1, n], \\
& \quad s_{i}^{t-1} + x_{i}^{t} = d_{i}^{2} + s_{i}^{t} \quad t \in [1, n], \\
& \quad s_{i}^{0} = s_{i}^{n} = 0 \quad i \in [1, 2], \\
& \quad x_{i}^{1} \leq (d_{i}^{1} + d_{i}^{2}) y_{i}^{1} \quad t \in [1, n], \\
& \quad x_{i}^{2} \leq d_{i}^{2} y_{i}^{2} \quad t \in [1, n], \\
& \quad y_{i}^{t} \in \{0, 1\} \quad t \in [1, n], i \in [1, 2], \\
& \quad x_{i}^{t} \geq 0 \quad t \in [1, n], i \in [1, 2], \\
& \quad s_{i}^{t} \geq 0 \quad t \in [1, n], i \in [1, 2].
\end{align*}
\]
Extreme Point Solutions

- $(i_1, i_2, j_1, j_2)$ is defined as a **regeneration interval** if for $i_1 \leq j_1 \leq j_2$, $s_{i_1-1}^1 = s_{i_2}^1 = s_{j_1-1}^2 = s_{j_2}^2 = 0$, $x_{i_1}^1 = d_{i_1i_2}^1 + d_{j_1j_2}^2$, or for $j_1 = j_2 + 1$, $s_{i_1-1}^1 = s_{i_2}^1 = 0$, $x_{i_1}^1 = d_{i_1i_2}^1$.

- $(j_1, j_2)$ is defined as a **regeneration subinterval** with $1 \leq j_1 \leq j_2 \leq n$ and $s_{j_1-1}^2 = s_{j_2}^2 = 0$, $x_{j_1}^2 = d_{j_1j_2}^2$.

(1, 3, 1, 5), (4, 4, 5, 4), (5, 6, 6, 6) are regeneration intervals. (3, 5), (1, 2), (6, 6) are regeneration subintervals.
Dynamic Programming Recursion

- \( G(i_2, j_2) \): the minimum cost of satisfying the demand in periods 1 to \( i_2 \) at the first echelon and the demand in periods 1 to \( j_2 \) at the second echelon, \( 0 \leq i_2 \leq j_2 \leq n \). \( G(0, 0) = 0 \) and \( G(0, k) = \infty \) if \( k \in [1, n] \).
- \( H(j_1, j_2) \): the minimum cost to satisfy the demand in periods \( j_1 \) to \( j_2 \) at the second echelon, \( 1 \leq j_1 \leq n + 1, 0 \leq j_2 \leq n \). \( H(j_1, j_2) = 0 \) if \( j_1 > j_2 \).

Forward DP recursion

For \( 1 \leq i_2 \leq j_2 \leq n \),

\[
G(i_2, j_2) = \min_{1 \leq i_1 \leq i_2} \{ G(i_1 - 1, j_1 - 1) + f_{i_1}^1 + c_{i_1}^1 d_{i_1 i_2}^1 + c_{i_1}^1 d_{j_1 j_2}^2 + H(j_1, j_2) \};
\]

for \( 1 \leq j_1 \leq j_2 \leq n \),

\[
H(j_1, j_2) = \min_{j_1 \leq j_3 \leq j_2} \{ H(j_1, j_3 - 1) + f_{j_3}^2 + c_{j_3}^2 d_{j_3 j_2}^2 \}.
\]

- Running time of DP: \( O(n^4) \).
- If \( d_{j}^1 = 0, \forall j \), the DP is equivalent to that of Melo and Wolsey (2010).
DP-based extended formulation (DPEF)

- DPEF is the dual of the LP model based on the DP recursions.
- Variables:
  - \( v_{i_1 i_2 j_1 j_2} = 1 \) if \((i_1, i_2, j_1, j_2)\) is a regeneration interval; otherwise, \( v_{i_1 i_2 j_1 j_2} = 0 \).
  - \( w_{j_1 j_3 j_2} = 1 \) if \((j_3, j_2)\) is a regeneration subinterval; otherwise, \( w_{j_1 j_3 j_2} = 0 \).

\[
\begin{align*}
(1,1) & \rightarrow (2,1) \rightarrow (3,1) \rightarrow (4,1) \rightarrow (5,1) \rightarrow (6,1) \\
(1,2) & \rightarrow (2,2) \rightarrow (3,2) \rightarrow (4,2) \rightarrow (5,2) \rightarrow (6,2)
\end{align*}
\]

\( v_{1315} = 1, v_{4454} = 1, v_{5666} = 1 \) and \( w_{135} = 1, w_{112} = 1, w_{666} = 1 \).

- DPEF is **tight** and **compact**: \( O(n^4) \) variables and \( O(n^3) \) constraints.
Valid Inequalities

Define $\beta(T, k)$ as the set of consecutive elements in set $T$ starting from $k$, where if $k \notin T$, $\beta(T, k) = \emptyset$.

Theorem (Zhang, K., Yaman, 2011)

For $0 \leq k \leq l \leq n$, let $T_1 \subseteq [1, k]$, $C \subseteq [1, k]$, $T_2 = C \cup [k + 1, l]$ and $T_3 \subseteq T_2$. Then the 2-echelon inequality

$$
\sum_{j \in [1,k] \setminus T_1} x_j^1 + \sum_{j \in T_1} \phi_j y_j^1 + \sum_{j \in T_2 \setminus T_3} x_j^2 + \sum_{j \in T_3} \psi_j y_j^2 \geq d_{1k}^1 + d_{1l}^2
$$

is valid for 2-ULS and facet-defining under certain conditions, where $\psi_j = \sum_{i \in \beta(T_2, j)} d_i^2$ and $\phi_j = d_{jk}^1 + d_{jl}^2 - \psi_j$.

2-echelon inequalities subsume $(\ell, S)$ inequalities of Barany et al. (1984)
e.g. For $n = 6$, $k = 4$, $l = 5$, $d^i_j = 1 \ \forall i = 1, 2, j = 1, \ldots, 6$, $x^1_1 + 6y^1_2 + x^1_3 + y^1_4 + x^2_2 + 2y^2_4 + x^2_5 \geq 9$ is a valid inequality with following choices of $T_1$, $T_2$ and $T_3$. 

![Diagram of valid inequalities]
Valid Inequalities

Proposition 1 (Zhang, K., Yaman, 2011)

Given a fractional point \((x^1, y^1, x^2, y^2) \in \mathbb{R}^{4n}\), there is an \(O(n^4)\) algorithm to find the most violated 2-echelon inequality, if any.

Separation:

- Shortest path problem on the network \(G = (V, A)\), for a given \(k\):
  \(V = \{1', 2, 2', \ldots, k, k', k + 1\}\) and \(A = \{(i', i + 1) : i = 1, \ldots, k\}\)
  \(\cup\{(i', (i + 1)') : i = 1, \ldots k\}\)
  \(\cup\{(i, j') : i \in [2, k - 2], j \in [i + 1, k - 1]\}\).

- Node \(i\) represents \(i \in T_2\), \(i'\) represents \(i' \not\in T_2\).

- For \(k = 4\), the shortest path network is:
Facility Location Formulation for 2-ULS

- **Variables:**
  - $z_{ut}^{11}$: the order quantity in period $u$ at the first echelon to satisfy the intermediate demand in period $t$, $1 \leq u \leq t \leq n$;
  - $z_{ut}^{12}$: the order quantity in period $u$ at the first echelon to satisfy the demand at the second echelon in period $t$, $1 \leq u \leq t \leq n$;
  - $z_{ut}^{22}$: the order quantity in period $u$ at the second echelon to satisfy the demand at the second echelon in period $t$, $1 \leq u \leq t \leq n$. 

Facility Location Formulation for 2-ULS

\[
\begin{align*}
\min & \quad \sum_{i=1}^{2} \sum_{t=1}^{n} (f_t^i y_t^i + c_t^i x_t^i) \\
\text{s.t.} & \quad \sum_{u=1}^{t} z_{ut}^{11} = d_t^1 \\
& \quad \sum_{u=1}^{t} z_{ut}^{22} = d_t^2 \\
& \quad \sum_{u=1}^{j} z_{ut}^{12} \geq \sum_{u=1}^{j} z_{ut}^{22} \\
& \quad z_{ut}^{11} \leq d_t^1 y_u^1 \\
& \quad z_{ut}^{12} \leq d_t^2 y_u^1 \\
& \quad z_{ut}^{22} \leq d_t^2 y_u^2 \\
& \quad x_t^1 = \sum_{u=t}^{n} (z_{tu}^{11} + z_{tu}^{12}) \\
& \quad x_t^2 = \sum_{u=t}^{n} z_{tu}^{22} \\
& \quad z_{ut}^{11}, z_{ut}^{12}, z_{ut}^{22} \geq 0 \\
y_t^i & \in \{0, 1\} \\
z_{ut}^{11}, z_{ut}^{12}, z_{ut}^{22} & \geq 0 \\
t \in [1, n], i \in [1, 2].
\end{align*}
\]
Facility Location Formulation for 2-ULS

- UFL formulation $\equiv$ multi-commodity formulation (Rardin and Wolsey, 1993) after projecting out the inventory variables.
- Without intermediate demands, constraints (10) and (11) are redundant. (Pochet and Wolsey, 2006)

**Proposition 2 (Zhang, K., Yaman, 2011)**

2-echelon inequalities can be obtained by projecting the UFL formulation onto the $(x^1, y^1, x^2, y^2)$ space.

**Proof idea**

Every valid projection inequality with 0-1 coefficients is a 2-echelon inequality.
Echelon Stock Formulation for 2-ULS

- **Variables:**
  - $e^1_t$: the total inventory at the first echelon at the end of period $t$, $t \in [1, n]$. Note that $e^1_t = s^1_t + s^2_t$.
  - $e^2_t$: the total inventory at the second echelon at the end of period $t$, $t \in [1, n]$. Note that $e^2_t = s^2_t$.

- **Echelon Stock Formulation:**

$$\min \sum_{i=1}^{2} \sum_{t=1}^{n} (f^i_t y^i_t + c^i_t x^i_t)$$

s.t. (5) – (8),

- $e^1_{t-1} + x^1_t = d^1_t + d^2_t + e^1_t$ \hspace{1cm} $t \in [1, n],$
- $e^2_{t-1} + x^2_t = d^2_t + e^2_t$ \hspace{1cm} $t \in [1, n],$
- $e^i_0 = e^i_n = 0$ \hspace{1cm} $i \in [1, 2],$
- $e^1_t \geq e^2_t$ \hspace{1cm} $t \in [1, n],$
- $e^i_t \geq 0$ \hspace{1cm} $t \in [1, n], i \in [1, 2]$. 


Echelon Stock Formulation for 2-ULS

Proposition (*Zhang, K., Yaman, 2011*)

The natural formulation with 2-echelon inequalities is stronger than the echelon stock reformulation with \((\ell, S')\) inequalities.
Hierarchy of Formulations

From stronger to weaker:

- Projection of the DP-based extended formulation (DPEF);
- Projection of the UFL formulation (or equivalently multi-commodity formulation);
- Natural formulation with 2-echelon inequalities;
- Echelon stock formulation with \((\ell, S)\) inequalities.
- Natural formulation.
Computational Results

- Tests on **multi-item** 2-ULS with a mode constraint.
- Environment: 1 GHz Dual-Core AMD Opteron(tm) Processor 1218 with 2GB RAM.
- Solver: IBM ILOG CPLEX 12.0 with an hour time limit.
- Data generation:
  - Horizon: $n \in \{60, 90, 120\}$,
  - Ratios of fixed and unit order costs: $\beta \in \{500, 1000, 2500\}$,
  - Numbers of items $r \in \{5, 10\}$,
  - Capacities (the maximum number of items ordered in each period) $\kappa \in \{2, 3, 5\}$.
- Average of five instances for each combination of $n, \beta, r, \kappa$. 
## Computational Results

<table>
<thead>
<tr>
<th>$n$.β.r.κ</th>
<th>MIP</th>
<th>UFL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>IGap</td>
<td>EGap</td>
</tr>
<tr>
<td>60.500.5.2</td>
<td>42.86</td>
<td>14.62</td>
</tr>
<tr>
<td>60.500.10.3</td>
<td>42.29</td>
<td>20.80</td>
</tr>
<tr>
<td>60.1000.5.3</td>
<td>40.11</td>
<td>17.13</td>
</tr>
<tr>
<td>60.2500.10.5</td>
<td>28.03</td>
<td>2.54</td>
</tr>
<tr>
<td>90.500.5.3</td>
<td>51.27</td>
<td>24.63</td>
</tr>
<tr>
<td>90.1000.5.3</td>
<td>49.39</td>
<td>31.43</td>
</tr>
<tr>
<td>90.1000.10.5</td>
<td>48.63</td>
<td>35.15</td>
</tr>
<tr>
<td>90.2500.10.3</td>
<td>41.05</td>
<td>33.16</td>
</tr>
<tr>
<td>120.500.10.3</td>
<td>55.15</td>
<td>30.54</td>
</tr>
<tr>
<td>120.1000.5.3</td>
<td>55.30</td>
<td>38.40</td>
</tr>
<tr>
<td>120.2500.5.2</td>
<td>49.03</td>
<td>42.49</td>
</tr>
<tr>
<td>120.2500.10.5</td>
<td>48.98</td>
<td>46.01</td>
</tr>
</tbody>
</table>
Concluding Remarks

- Size matters. (DPEF is too large.)
- UFL formulation is not tight in general, but extremely strong in practice.
- Complete linear description of 2-ULS in the original space is open (even with zero intermediate demands)
- Complete characterization of the UFL projection inequalities is open.
- Automatic reformulation ideas (van Roy and Wolsey, 1987)
Thanks!