Cutting Planes for Stochastic Integer Programs

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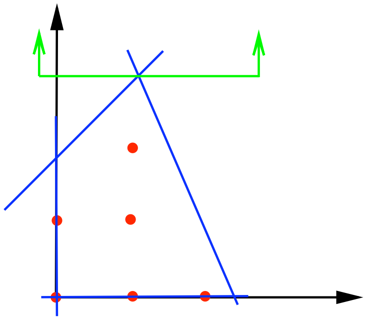
- Stochastic lot-sizing problems.
- Chance constrained stochastic programs (stochastic RHS)
- Computational Results

Cutting Planes for IP

Cutting Planes for IPs

$$\min\{cx : Ax \geq b, \; x \in \mathbb{Z}^n_+\}$$



LP Relaxation

Cutting Planes for IPs

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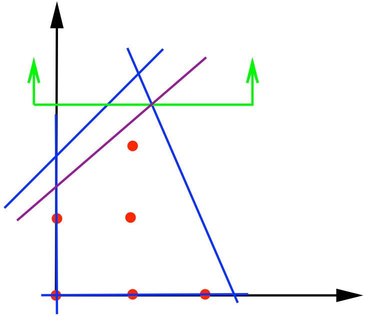

A cutting plane

Cutting Planes for IPs

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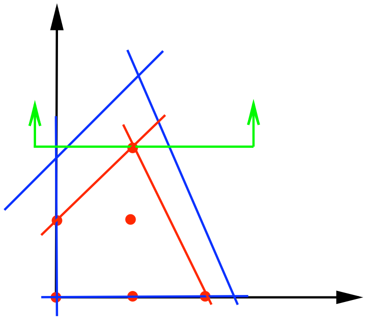

Facets of convex hull

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The (Cover) cut

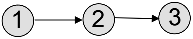
$$\sum_{j \in C} x_j \leq |C| - 1$$

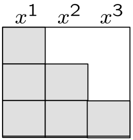
where C is s.t. $\sum_{j \in C} a_{ij} > b_i$ is valid for X_i and hence valid for X.

Multi-Stage Stochastic IP

Multiperiod IP

$$(P): \quad \begin{array}{lll} \min & \sum_{t=1}^{T} c_t x_t & \\ \text{s.t.} & \sum_{\tau=1}^{t} A_{\tau t} x_\tau \geq b_t & \forall\ t = 1, \ldots, T \\ & x_t \in X_t & \forall\ t = 1, \ldots, T \end{array}$$





Stochastic Extension

- The parameters $\{c_t, A_{\cdot t}, b_t\}_{t=2}^{T}$ are uncertain and evolve according to a finite scenario tree with T stages.
  - The nodes on a path $\mathcal{P}(i)$ from the root to a node i in stage t_i represents a joint realization of $\{c_t, A_{\cdot t}, b_t\}_{t=1}^{t_i}$ and has probability p_i.

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- The decision process:

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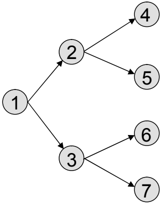
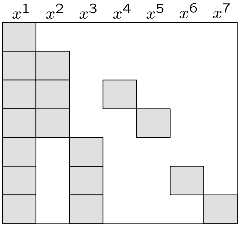
- Minimize expected total cost:

$$\min_{x_1} c_1 x_1 + \mathbb{E}\big[\min_{x_2} c_2 x_2 + \mathbb{E}[\min_{x_3} c_3 x_3 + \mathbb{E}[\ldots]]\big]$$

Multistage Stochastic IP

$$(SIP): \quad \begin{array}{lll} \min & \sum_{i \in \mathcal{T}} p_i c_i x_i & \\ \text{s.t.} & \sum_{j \in \mathcal{P}(i)} A_{ij} x_j \geq b_i & \forall\, i \in \mathcal{T} \\ & x_i \in X_i & \forall\, i \in \mathcal{T} \end{array}$$





Example: Lot-Sizing

- Eliminate inventory variables.
- Deterministic:

$$\begin{array}{lll} \min & \sum_{t=1}^{T}(\alpha_t y_t + \beta_t x_t) + \gamma s_0 & \\ \text{s.t.} & s_0 + \sum_{\tau=1}^{t} y_\tau \geq \sum_{\tau=1}^{t} d_\tau & \forall\, t = 1, \ldots, T \\ & 0 \leq y_t \leq a_t x_t, \;\; x_t \in \{0, 1\} & \forall\, t = 1, \ldots, T \end{array}$$

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- Stochastic:

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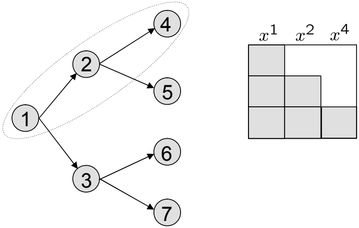
Path Sets

- Let $\mathcal{X}_{\mathcal{T}}$ denote the feasible region of SIP. Call it a tree set.
- Given a node i, the path set $\mathcal{X}_i$ (corresponding to node i) is defined by the constraints corresponding to the nodes $j \in \mathcal{P}(i)$.

$$\mathcal{X}_i := \left\{ \{x_i\}_{i \in \mathcal{T}} : \sum_{k \in \mathcal{P}(j)} A_{jk} x_k \geq b_j,\ x_j \in X_j,\ \forall j \in \mathcal{P}(i) \right\}$$

Path Set (contd)



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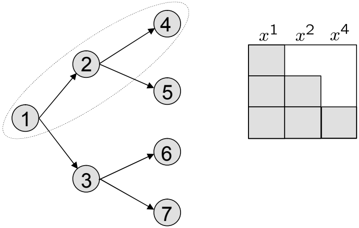

- $\mathcal{X}_i$ represents a deterministic t_i-period model.
- $\mathcal{X}_i \supseteq \mathcal{X}_{\mathcal{T}} \Rightarrow$ Deterministic cuts can be used.
- Given (deterministic) cuts for path sets, how to "combine" these to get a cut for the tree set?

Example

- Consider the following simple two-stage stochastic IP with two scenarios:

$$
\begin{array}{llll}
y & +\,5x_1 & & \geq 2 \\
y & & +\,5x_2 & \geq 5 \\
y, & x_1, & x_2 & \geq 0 \quad \text{and integer.}
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- But these cuts are not enough for $\text{conv}(\mathcal{X}_\mathcal{T})$ since $\text{conv}(\mathcal{X}_\mathcal{T} := \mathcal{X}_1 \cap \mathcal{X}_2) \subset \text{conv}(\mathcal{X}_1) \cap \text{conv}(\mathcal{X}_2)$.

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- The path cuts admit the fractional point $(y = 2, x_1 = 0, x_2 = 3/5) \notin \text{conv}(\mathcal{X}_\mathcal{T})$.

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- The path cuts $y + 2x_1 \geq 2$ and $y + 5x_2 \geq 5$ can be combined to obtain the tree cut

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- Cuts off the fractional point $(y = 2, x_1 = 0, x_2 = 3/5)$.
- In fact, conv$(\mathcal{X}_\mathcal{T})$ is given by
$$\begin{array}{llll} y & +2x_1 & & \geq 2 \\ y & & +5x_2 & \geq 5 \\ y & +2x_1 & +3x_2 & \geq 5 \\ y, & x_1, & x_2 & \geq 0. \end{array}$$

A simple combination scheme: Pairing

The pairing cut

Notation: Given two vectors $a_1, a_2 \in \mathbb{R}^n$, $\min(a_1, a_2)$ and $\max(a_1, a_2)$ are component-wise. Given a vector a and a scalar b, $a + b := a + b\mathbf{1}$ and $\min\{a, b\} := \min\{a, b\mathbf{1}\}$ where $\mathbf{1}$ is a vector of ones.

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Theorem

Suppose the cuts

$$g_1 y + a_1 x \geq b_1 \text{ and } g_2 y + a_2 x \geq b_2$$

with $b_1 \leq b_2$ are valid for $(y, x) \in X \subset \mathbb{R}^p_+ \times \mathbb{Z}^n_+$, then the pairing cut:

$$\varphi y + \phi x \geq b_2$$

is valid for X, where
$\varphi = \max\{g_1, g_2\}$ *and* $\phi = \min\{a_1 + (b_2 - b_1), \max\{a_1, a_2\}\}$.

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- From the pairing theorem $\varphi = \max\{1,1\} = 1$ and

$$\begin{aligned}\phi[1] &= \min\{a_1[1] + (b_2 - b_1), \max\{a_1[1], a_2[1]\}\} \\ &= \min\{2 + (3), \max\{2,0\}\} = 2\end{aligned}$$

$$\begin{aligned}\phi[2] &= \min\{a_1[2] + (b_2 - b_1), \max\{a_1[2], a_2[2]\}\} \\ &= \min\{0 + (3), \max\{0,5\}\} = 3\end{aligned}$$

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- Thus we obtain the pairing inequality:

$$y + 2x_1 + 3x_2 \geq 5.$$

Remarks

- The pairing cut is a *split cut* (Cook et al.'90) based on the disjunction
$$\sum_{j \in J} x_j \leq 0 \ \wedge \sum_{j \in J} x_j \geq 1$$
where $J = \{j : a_{1j} + b_2 - b_1 < \max(a_{1j}, a_{2j})\}$
- In case of non-negative coefficients, can be obtained by *mixing* (Günlük & Pochet'01) mixed-integer rounding cuts.

Sequential pairing

- Given K cuts
$$g_i y + a_i x \geq b_i \quad i = 1, \ldots, K$$
indexed such that $b_1 \leq b_2 \leq \cdots \leq b_K$.
- We can get different cuts by repeated application of pairing. The pairing order is important.
- Sequential pairing: Pairing in the order of increasing rhs values, i.e, 1 to K.

From Paths to Trees: Generating cuts for Stochastic IPs

Notation

Tree Set:

$$\mathcal{X}_{\mathcal{T}} = \left\{ (y_i, x_i)_{i \in \mathcal{T}} : \sum_{j \in \mathcal{P}(i)} (G_{ij} y_j + A_{ij} x_j) \geq b_i,\ y_i \in \mathbb{R}^p_+,\ x_i \in \mathbb{Z}^n_+\ \forall\, i \in \mathcal{T} \right\}$$

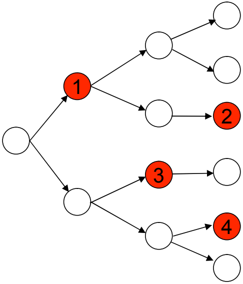
Path Set:

$$\mathcal{X}_i = \left\{ (y_j, x_j)_{j \in \mathcal{T}} : \sum_{k \in \mathcal{P}(j)} (G_{jk} y_k + A_{jk} x_k) \geq b_j,\ y_j \in \mathbb{R}^p_+,\ x_j \in \mathbb{Z}^n_+\ \forall\, j \in \mathcal{P}(i) \right\}$$

Notation

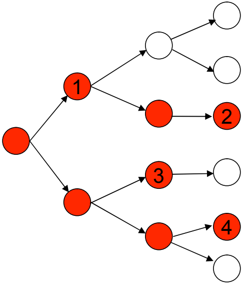
- For $j \in \mathcal{T}$, let $\mathcal{T}(j)$ denote the set of nodes that are descendants of j and include j.
- Given $R \subseteq \mathcal{T}$, define
$$\mathcal{T}_R = \cup_{i \in R} \mathcal{P}(i),$$
i.e., the subtree induced by R.
- Given $R \subseteq \mathcal{T}$ and $j \in \mathcal{T}$, define
$$R(j) = R \cap \mathcal{T}(j).$$

Notation



$$R = \{1, 2, 3, 4\}.$$

Notation



$\mathcal{T}_R$ = all red nodes. $R(1) = \{1, 2\}$.

The Tree Cut

Given a subset of nodes $R = \{1, 2, \ldots, K\} \subseteq \mathcal{T}$, suppose that the path cuts

$$\sum_{j \in \mathcal{P}(i)} (g_{ij} y_j + a_{ij} x_j) \geq b_i$$

are valid for the path sets $\mathcal{X}_i$ for all $i \in R$, and are such that $g_{ij} \in \mathbb{R}^p_+$, $a_{ij} \in \mathbb{R}^n_+$ and

$$b_1 \leq b_2 \leq \cdots \leq b_K.$$

The Tree Cut

Theorem

The following tree cut is valid for the tree set $\mathcal{X}_{\mathcal{T}}$:

$$\sum_{j \in \mathcal{T}_R} (\varphi_j(R) y_j + \phi_j(R) x_j) \geq b_K$$

where

$$\varphi_j(R) = \max_{i \in R}\{g_{ij}\} \text{ and } \phi_j(R) = \min\{\max_{i \in R}\{a_{ij}\}, \sum_{k \in R(j)} (b_k - b_{k-1})\},$$

with $b_0 := 0$.

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with $b_0 := 0$.

Proof: Sequential pairing of the path cuts 1 through K.

Stochastic lot-sizing

Stochastic Lot-sizing

- The stochastic lot-sizing set is

$$\mathcal{X}_{\text{SLP}} = \Big\{ (s_0, y, x) \in \mathbb{R}_+ \times \mathbb{R}_+^{|\mathcal{T}|} \times \{0,1\}^{|\mathcal{T}|} : s_0 + \sum_{j \in \mathcal{P}(i)} y_j \geq d_{0i},\ y_i \leq a_i x_i,\ i \in \mathcal{T} \Big\}.$$

SLS: The tree inequalities

Theorem

Given a subset $R = \{1, 2, \ldots, K\} \subseteq \mathcal{T}$ *, such that* $b_1 \leq \cdots \leq b_K$ *where* $b_i = \max_{j \in \mathcal{P}(i)} \{d_{0j}\}$*, and a subset* $S_R \subseteq \mathcal{T}_R$*, the following inequality is valid for* $\mathcal{X}_{\text{SLP}}$

$$s_0 + \sum_{j \in S_R} y_j + \sum_{j \in \overline{S}_R} \phi_j(R) x_j \geq b_{i_K},$$

where $\overline{S}_R = \mathcal{T}_R \setminus S_R$ *and* $\phi_j(R) = \min\{a_j, \sum_{k \in R(j)} (b_k - b_{k-1})\}$ *with* $b_0 = 0$*.*

SLS: Separation

Since the tree inequality depends on the choice of R and S_R, we refer to it as a (R, S_R) inequality ... exponentially many.

Theorem

(R, S_R) inequalities for a fixed set R can be separated in $(|R| \log |\mathcal{T}|)$ time.

We do not have a polynomial scheme for separating general (R, S_R) inequalities. We use a heuristic separation scheme.

SLS: The uncapacitated case

Theorem

There is a set of necessary and sufficient conditions under which an (R, S_R) inequality is facet-defining for the stochastic uncapacitated lot-sizing problem (ULSP).

Theorem

The (R, S_R) inequalities define the convex hull for 2-stage stochastic ULSP problems.

Chance Constrained SP

Chance-constrained SP with random RHS

- Formulation:

$$\begin{array}{ll} \min_x & cx \\ \text{s.t.} & x \in X \\ & \Pr\{Ax \geq \tilde{b}\} \geq 1 - \epsilon. \end{array}$$

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- Rewrite as:

$$\begin{array}{ll} \min_{x,y} & cx \\ \text{s.t.} & x \in X,\ y = Ax \\ & \Pr\{y_i \geq \tilde{b}_i\ i = 1, \ldots, m\} \geq 1 - \epsilon. \end{array}$$

MIP Formulation

- Assume $\Pr\{\tilde{b} = b^k\} = p_k$ for $k = 1, \ldots, K$.
- Also (w.l.o.g) $b_k \geq 0$ for all k.

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- MIP formulation

$$\begin{array}{ll} \min_{x,y,z} & cx \\ \text{s.t.} & x \in X,\ y = Ax \\ & y_i + Mz_k \geq b_i^k \quad i = 1, \ldots, m,\ k = 1, \ldots, K \\ & \sum_{k=1}^{K} p_k z_k \leq \epsilon \\ & y_i \geq 0,\ z_k \in \{0, 1\} \quad i = 1, \ldots, m,\ k = 1, \ldots, K. \end{array}$$

  where $M \geq b_i^k$ for all i and k.
- $z_k = 1 \Rightarrow$ scenario k may be violated.

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where $M \geq b_i^k$ for all i and k.

- $z_k = 1 \Rightarrow$ scenario k may be violated.
- This MIP is very hard due to weak LP relaxation.

Single row relaxation

- Consider a single row system.

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- For any $R = \{k_1, \ldots, k_R\} \subseteq \{1, \ldots, K\}$ the tree inequality is

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- The tree inequalities for all R describe the convex hull. Moreover these can be separated efficiently (Atamturk et al '01).

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- Cannot have $z_1 = z_2 = \cdots = z_{\ell+1} = 1 \quad \Rightarrow z_{\ell+1} = 0 \quad \Rightarrow y \geq b^{\ell+1}$
- Replace

$$y + b^i z_i \geq b^i \quad i = 1, \ldots, K$$

  with

$$y + (b^i - b^{\ell+1}) z_i \geq b^i \quad i = 1, \ldots, \ell$$

- Stronger and more compact
- Now get tree cuts for this set $\Rightarrow$ facet-defining

Computational Results

Small 2-stage stochastic IPs

Random instances of

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{m} h_i y_i + \sum_{s=1}^{S} p_s \Big(\sum_{j=1}^{n} c_j^s x_j^s \Big) & \\
& \sum_{i=1}^{m} g_i y_i + \sum_{j=1}^{n} a_j^s x_j^s \geq b_s & s = 1, \ldots, S \\
& x_j^s \in \{0, 1\} & j = 1, \ldots, n, s = 1, \ldots, S \\
& y_i \geq 0 & i = 1, \ldots, m.
\end{aligned}
$$

			$S = 10$		$S = 20$		$S = 40$	
m	n		OptVal	Gap	OptVal	Gap	OptVal	Gap
2	1	LP	685.99	8.36%	388.66	5.96%	497.25	7.69%
		LP+CUTS	693.98	7.29%	400.19	3.17%	522.35	3.03%
		IP	748.58		413.29		538.67	
		# CUTS	34		74		341	
2	2	LP	507.76	33.72%	356.65	14.23%	464.42	13.56%
		LP+CUTS	710.69	7.22%	415.34	0.12%	529.88	1.38%
		IP	766.03		415.83		537.30	
		# CUTS	39		119		530	
2	3	LP	400.18	20.50%	339.61	12.71%	437.98	13.60%
		LP+CUTS	448.58	10.89%	357.13	8.21%	470.21	7.25%
		IP	503.38		389.06		506.95	
		# CUTS	20		63		347	
3	1	LP	533.31	2.84%	285.17	2.09%	387.16	4.80%
		LP+CUTS	542.35	1.19%	291.25	0.00%	406.67	0.00%
		IP	548.88		291.25		406.67	
		# CUTS	25		34		173	
3	2	LP	433.80	21.02%	280	3.86%	375.75	6.84%
		LP+CUTS	540.33	1.63%	291.23	0.00%	403.35	0.00%
		IP	549.27		291.23		403.35	
		# CUTS	45		59		289	
3	3	LP	420.45	14.19%	279.87	4.38%	360.44	9.80%
		LP+CUTS	459.61	6.19%	292.69	0.00%	399.61	0.00%
		IP	489.95		292.69		399.61	
		# CUTS	24		67		375	

Uncapacitated lot-sizing

Random instances of

$$\begin{aligned} \min \quad & \sum_{i \in \mathcal{T}} p_i(\alpha_i y_i + \beta_i x_i + h_i s_i) \\ \text{s.t.} \quad & s_{a(i)} + y_i = d_i + s_i \;\; \forall\, i \in \mathcal{T} \\ & 0 \le y_i \le M_i x_i, \;\; x_i \in \{0, 1\} \;\; \forall\, i \in \mathcal{T} \end{aligned}$$

with a binary scenario tree.

T	β/h	α/h	LP Gap %	$\|R\| = 1$	$\|R\| = 2$	General R
10	3500	50	13.06	0.11	0.01	0.01
				3424	2513	51
10	3500	100	12.10	0.10	0.01	0.01
				3374	2630	80
10	3500	200	9.87	0.08	0.00	0.00
				3433	1868	12
10	7000	50	22.13	0.19	0.02	0.01
				3183	4267	98
10	7000	100	20.81	0.26	0.02	0.01
				3420	3679	84
10	7000	200	17.35	0.35	0.09	0.02
				3238	4718	310
11	3500	50	5.25	0.06	0.02	0.01
				7691	6675	291
11	3500	100	4.99	0.04	0.00	0.00
				7769	5177	125
11	3500	200	4.36	0.03	0.00	0.00
				7911	3204	24
11	7000	50	9.57	0.16	0.02	0.02
				7179	12042	280
11	7000	100	9.21	0.16	0.02	0.02
				7437	9968	223
11	7000	200	8.17	0.11	0.01	0.01
				7656	7452	71

Root node gaps

T	β/h	α/h	No. of cuts	Optimality gap %	Nodes	CPU secs
10	3500	50	612	0.00	131850	220.2
			5996	0.00	0	3.0
10	3500	100	598	0.00	39828	70.8
			6129	0.00	0	5.4
10	3500	200	513	0.00	343	2.4
			5313	0.00	0	1.8
10	7000	50	671	0.00	1336827	2619.7
			7737	0.00	0	13.9
10	7000	100	682	0.00	915006	1715.7
			7213	0.00	0	5.0
10	7000	200	597	0.00	13124	26.0
			8407	0.00	0	23.5
11	3500	50	1065	0.00	644407	820.2
			14946	0.00	0	63.5
11	3500	100	994	0.00	9807	42.9
			13071	0.00	0	3.3
11	3500	200	852	0.00	889	9.2
			11139	0.00	0	2.5
11	7000	50	1126	0.03[3]	826644	***
			20784	0.00	0	189.0
11	7000	100	1112	0.03[3]	907471	***
			17796	0.00	0	35.9
11	7000	200	1084	0.00	414122	1496.7
			15179	0.00	0	15.5

Branch and cut

Chance-Constrained Transportation Problems

- Randomly generated instances of:

$$\begin{array}{ll} \min & \sum_i \sum_j c_{ij} x_{ij} \\ \text{s.t.} & \sum_j x_{ij} \leq s_i \quad \forall\, i \\ & \Pr\{\sum_i x_{ij} \geq \tilde{d}_j \quad \forall\, j\} \geq 1 - \epsilon \\ & x_{ij} \geq 0 \quad \forall\, i,\ j. \end{array}$$

- Number of customers $m \in \{100, 200\}$.
- Time limit = 1 hr.
- Number of equiprobable scenarios $K \in \{1000, 2000, 3000\}$.
- Results are averages over 5 instances.

Results

ϵ	m	N	MIP Gap	+Tree Cuts Time(s)
0.05	100	1000	0.18%	7.7
	100	2000	1.31%	31.8
	200	2000	1.02%	61.4
	200	3000	2.56%	108.6
0.10	100	1000	2.24%	34.6
	100	2000	5.12%	211.3
	200	2000	4.69%	268.5
	200	3000	6.20%	812.7

Conclusions

- Valid cuts from deterministic IPs can be “combined” to obtain useful cuts for stochastic counterparts.
- We used a very simple combination scheme - allows closed form cuts.
- Exploits differences of RHS values across scenarios.
- Decomposition structure is destroyed.
- More general combination schemes, e.g. mixing, may be investigated.
- Applications in other classes of stochastic IPs.