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Convex Relaxations and MIQP Reformulation for Quadratic Programs with Probabilistic Constraints

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Problem formulation

- Consider quadratic program with a probabilistic (or chance) constraint:

$$
\begin{aligned}
(P) \qquad & \min\ x^T Q x + c^T x \\
& \text{s.t. } x \in X, \\
& \qquad \mathbb{P}(\xi^T B x \geq R) \geq 1 - \epsilon,
\end{aligned}
$$

  where

$$X = \{x \mid Ex \leq f,\ 0 \leq x \leq u,\ x^T A_i x + b_i^T x + d_i \leq 0,\ i = 1, \ldots, r\},$$

  Q and A_i, $i = 1, \ldots, r$ are positive semidefinite $n \times n$ symmetric matrices, $c \in \Re^n$, B is an $m \times n$ matrix, ξ is a random vector in $\Re^m$, $\mathbb{P}$ denotes the probability, $0 < \epsilon < 1$.
- Difficulty: nonconvexity of the feasible set. (P) is in general NP-hard.

Application: VaR constrained portfolio selection

- Let ξ be the random vector of expected returns of n risky assets. Suppose ξ has the following mean vectors:

$$\mu = (\mu_1, \ldots, \mu_n)^T, \quad \mu_i = E(\xi_i),\ i = 1, \ldots, n,$$

and co-variance matrix:

$$\Sigma = E[(\xi - E(\mu))(\xi - E(\mu))^T].$$

- The Value-at-Risk (VaR) constraint can be expressed as

$$\mathbb{P}(\xi^T x \geq R) \geq 1 - \epsilon,$$

where $\xi^T x$ represents the random return of the portfolio with weight vector x, R is the prescribed minimal level of return, and ϵ is usually a small number, $\epsilon = 0.05$, for example.

- The VaR-constrained mean-variance portfolio selection model:

$$\begin{aligned} \min \ & x^T \Sigma x - \sigma \mu^T x \\ \text{s.t.}\ & \mathbb{P}(x^T \xi \geq R) \geq 1 - \epsilon, \\ & Ex \leq f,\ 0 \leq x \leq u. \end{aligned}$$

- VaR-constrained mean-variance problem using two multifactor risk models:

$$\begin{aligned} \min \quad & \frac{1}{2} x^\top B_1^\top B_1 x + x^\top D_1 x, \\ \text{s.t.} \quad & \mathbb{P}(x^T \xi \geq R) \geq 1 - \epsilon, \\ & x^\top B_2^\top B_2 x + x^\top D_2 x \leq \sigma_0, \\ & Ex \leq f,\ 0 \leq x \leq u. \end{aligned}$$

Literature review

- Extensive study for LP with a special probabilistic constraint: $\mathbb{P}(Ax \geq \xi) \geq 1 - \epsilon$, where ξ is a random vector, Prékopa (2003), Ruszczynski (2002), Luedtke, Ahmed and Nemhauser (2010) ...
- If the random vector ξ has a known (continuous) distribution, then safe (conservative) approximation technique can be used to obtain a convex approximation, Nemirovski and Shapiro (2006).
- Scenario approximation is another way of constructing tractable convex approximations to probabilistic constraint. Lower bounds of sample size to ensure the feasibility of the solution generated from scenario approximations are derived in Calafiore and Campi (2005, 2006) and Nemirovski and Shapiro (2009).

- In this talk, we focus on the case when ξ has a *finite discrete distribution*: ξ takes finite number of values $\xi^1, \ldots, \xi^N \in \Re^m$ with equal probability, called scenarios.
- Let α_i be the minimum value of $\xi_i^T Bx$ for all possible scenarios, i.e, $(\xi^i)^T Bx \geq \alpha_i$, $i = 1, \ldots, N$. Let $K = \lfloor N\epsilon \rfloor$.
- Then, (P) can be reformulated as a mixed integer QP program (standard MIQP):

$$
(\mathrm{MIQP}_0) \qquad \begin{aligned} \min\ & x^T Qx + c^T x \\ \text{s.t. } & (\xi^i)^T Bx \geq R + y_i(\alpha_i - R),\ i = 1, \ldots, N, \\ & \sum_{i=1}^{T} y_i \leq K, \\ & x \in X,\ y_i \in \{0, 1\},\ i = 1, \ldots, N. \end{aligned}
$$

- Convex mixed integer QCQP problem can be solved by branch-and-bound method based on the continuous relaxation. For example, MIQCP solver in CPLEX 12.1 is able to solve convex MIQP problems using branch-and-bound method in which the lower bounds are computed by continuous relaxation or LP relaxation at each node of the search tree.
- Research questions: *Can we do better than the standard MIQP reformulation*? More precisely:
  - Does there exist tighter convex relaxations of (P) than the continuous relaxation of (MIQP_0)?
  - Does there exist a more efficient reformulation of mixed integer QCQP for (P) than (MIQP_0)?

Outline

- Convex relaxations via Lagrangian decomposition
- An improved MIQP reformulation
- Preliminary computational results
- Conclusions

Lagrangian decomposition

- Define

$$\begin{aligned}\alpha_i &= \min_{x\in X}\ (\xi^i)^T Bx,\ \ i=1,\dots,N\\ \beta_i &= \max_{x\in X}\ (\xi^i)^T Bx,\ \ i=1,\dots,N\\ \Theta &= \{\theta\in\Re^N \mid Q-\sum_{i=1}^N \theta_i B^T\xi^i(\xi^i)^T B\succeq 0\}.\end{aligned}$$

- For any $\theta\in\Theta$, problem (P) can be written as

$$\begin{aligned}(P_\theta)\qquad \min\ & x^T(Q-\sum_{i=1}^N \theta_i B^T\xi^i(\xi^i)^T B)x + c^T x + \sum_{i=1}^N \theta_i v_i^2\\ \text{s.t. }\ & v_i=(\xi^i)^T Bx,\quad i=1,\dots,N,\quad \text{(link constraint)}\\ & v_i\ge R+y_i(\alpha_i-R),\ \ i=1,\dots,N,\\ & e^T y\le K\\ & x\in X,\ \ \alpha\le v\le\beta,\ \ y\in\{0,1\}^N.\end{aligned}$$

- Associating a multiplier λ_i to the link constraint $v_i = (\xi^i)^T Bx$, we have the following Lagrangian relaxation problem of (P):

$$d(\lambda) = d_1(\lambda) + d_2(\lambda),$$

where

$$\begin{aligned} d_1(\lambda) &= \min\ x^T(Q - \sum_{i=1}^{N} \theta_i B^T \xi^i (\xi^i)^T B)x + (c - \sum_{i=1}^{N} \lambda_i B^T \xi^i)^T x \\ & \quad \text{s.t. } x \in X \\ d_2(\lambda) &= \min\ \sum_{i=1}^{N} \theta_i v_i^2 + \lambda_i v_i \\ & \quad \text{s.t. } v_i \geq R + y_i(\alpha_i - R),\ y_i \in \{0,1\},\ i = 1,\ldots,N, \\ & \qquad e^T y \leq K,\ \alpha \leq v \leq \beta. \end{aligned}$$

- $d_1(\lambda)$ and $d_2(\lambda)$ are two easy subproblems!

- The Lagrangian dual of (P_θ) is

$$(D_\theta) \quad \max_{\lambda} \; d(\lambda)$$

- The best θ can be found via the following program

$$(D) \quad \max_{\theta \in \Theta} \; v(D_\theta)$$

- We can show that
  - (D_θ) can be reduced to an SOCP problem (for fixed $\theta \in \Theta$).
  - (D_θ) is tighter than (or at least as tight as) the continuous relaxation of (MIQP_0) for any fixed $\theta \in \Theta$ and $\theta \geq 0$.
  - (D_θ) is equivalent to the continuous relaxation of a new reformulation of (P).
  - (D) can be reduced to an SDP problem (best bound for all admissible θ).

SOCP relaxation

For any $\theta \in \Theta$, (D_θ) can be reduced to (DSOCP_θ):

$$
\begin{aligned}
\max\ & \tau + t + e^T\rho \\
\text{s.t.}\ & -t - Ks - e^Tz \geq 0, \\
& -z_i - s \leq -\theta_i R^2 - \lambda_i R + \delta_i,\ i = 1, \ldots, N, \\
& \begin{pmatrix} 2\theta_i + 2\mu_i & \lambda_i - \mu_i(\alpha_i + R) \\ \lambda_i - \mu_i(\alpha_i + R) & 2\mu_i\alpha_i R - 2\delta_i \end{pmatrix} \succeq 0,\ i = 1, \ldots, N, \\
& \begin{pmatrix} 2\theta_i + 2\sigma_i & \lambda_i - \sigma_i(R + \beta_i) \\ \lambda_i - \sigma_i(R + \beta_i) & 2\sigma_i R\beta_i - 2\rho_i \end{pmatrix} \succeq 0,\ i = 1, \ldots, N, \\
& \begin{pmatrix} 2\tilde{Q} & \tilde{c} \\ \tilde{c}^T & 2\tilde{\tau} \end{pmatrix} \succeq 0, \\
& (z, \mu, \sigma, \zeta, \eta, \vartheta, \pi) \geq 0.
\end{aligned}
$$

- The conic dual of (DSOCP_θ) is (SOCP_θ):

$$
\begin{aligned}
\min\ & x^T(Q - \sum_{i=1}^{N} \theta_i B^T \xi^i (\xi^i)^T B)x + c^T x \\
& \quad + \theta^T(\phi + \varphi) - R^2 \theta^T y \\
\text{s.t. } & x \in X, \\
& (\xi^i)^T Bx = w_i + z_i - y_i R, i = 1, \ldots, N, \\
& e^T y = K, y \in [0,1]^p, \\
& \phi_i - (\alpha_i + R) z_i + \alpha_i R y_i \le 0, \\
& \phi_i y_i \ge z_i^2, \phi_i \ge 0,\ i = 1, \ldots, N,\ (\text{SOCP constraints}) \\
& \varphi_i - (\beta_i + R) w_i + R\beta_i \le 0, \varphi_i \ge w_i^2,\ i = 1, \ldots, N.
\end{aligned}
$$

- $v(\mathrm{SOCP}_\theta) \ge v(\mathrm{QP})$, where (QP) is the continuous relaxation of (MIQP_0).

- In particular, when $\theta \geq 0$, (SOCP_θ) is equivalent to:

$$
\begin{aligned}
\min\ & x^T(Q - \sum_{i=1}^{N} \theta_i B^T \xi^i (\xi^i)^T B)x + w^T \mathrm{diag}(\theta) w + c^T x \\
& + \theta^T(\phi - R^2 y) \\
\text{s.t. } & x \in X, \\
& (\xi^i)^T B x = w_i + z_i - y_i R,\ \ i = 1, \dots, N, \\
& e^T y = K, y \in [0,1]^p, \\
& \alpha_i y_i \leq z_i \leq R y_i, \\
& \phi_i y_i \geq z_i^2,\ \phi_i \geq 0,\ \ i = 1, \dots, N,\ \ (\text{SOCP constraints}) \\
& R \leq w \leq \beta.
\end{aligned}
$$

- This reformulation can be interpreted as adding perspective cut or using the convex envelope of a separable function on some semi-continuous variables after reformulating the original problem.

MIQP reformulation

- This formulation also suggest that (SOCP_θ) is a continuous relaxation of the mixed-integer quadratic program:

$$
\begin{aligned}
\min\ & x^T(Q - \sum_{i=1}^{N} \theta_i B^T \xi^i (\xi^i)^T B)x + w^T \mathrm{diag}(\theta) w + c^T x \\
& + \theta^T(\phi - R^2 y) \\
\mathrm{s.t.}\ & x \in X, \\
& (\xi^i)^T Bx = w_i + z_i - y_i R, i = 1, \ldots, N, \\
& e^T y = K, y \in \{0,1\}^p, \\
& \alpha_i y_i \leq z_i \leq R y_i, \\
& \phi_i y_i \geq z_i^2, \phi_i \geq 0,\ i = 1, \ldots, N,\ (\mathrm{SOCP\ constraints}) \\
& R \leq w \leq \beta.
\end{aligned}
$$

- The continuous relaxation of (MIQP_θ) is exactly (SOCP_θ).
- For any $\theta \in \Re_+^N \cap \Theta$, it holds $v(\mathrm{MIQP}_\theta) = v(\mathrm{MIQP}_0)$.

How to choose the parameter θ?

- A natural way is to choose θ such that (MIQP_θ) has the tightest continuous bound among all admissible θ.
- This leads to solve the following problem:

$$(\mathrm{SDP_l}) \qquad \max\ \{v(\mathrm{SOCP}_\theta) \mid \theta \geq 0,\ \theta \in \Theta\},$$

  which can be reduced to an SDP problem.
- Intuition: $\theta = 0 \Rightarrow$ standard MIQP. So we may choose θ "*as large as possible*." This leads to a heuristic and simple SDP for choosing θ

$$(\mathrm{SDP_s}) \qquad \max\ \{e^T\theta \mid \theta \geq 0,\ \theta \in \Theta\}.$$

- When Q is positive definite and θ is set to be $\theta = te$, we can get a simple formulation of the optimal solution to $(\mathrm{SDP_s})$:

$$t^* = \frac{1}{\lambda_{\max}(U^T B^T \Gamma\Gamma^T BU)}, \tag{1}$$

where $\Gamma = (\xi^1, \ldots, \xi^N)$.

Numerical results

- Purpose:
  - Comparing the new SOCP and SDP bounds with the QP bound.
  - Comparing the new reformulation (MIQP_θ) with the standard reformulation (MIQP_0).
- We use 3 choices of the parameter θ:
  - $(\mathrm{MIQP}^{\mathrm{l}}_\theta)$: the MIQP reformulation with θ computed by solving $(\mathrm{SDP}_{\mathrm{l}})$;
  - $(\mathrm{MIQP}^{\mathrm{s}}_\theta)$: the MIQP reformulation with θ computed by solving $(\mathrm{SDP}_{\mathrm{s}})$;
  - $(\mathrm{MIQP}^{\mathrm{e}}_\theta)$: the MIQP reformulation with θ computed by the formulation (1);

Test Problems

- Test problems:

$$(\mathrm{MV_p}) \qquad \begin{aligned} & \min \ x^T \Sigma x \\ & \text{s.t. } \mathbb{P}(x^T \xi \geq R) \geq 1 - \epsilon, \\ & \qquad e^T x = 1, \ 0 \leq x \leq 0.5e. \end{aligned}$$

- To generate the return scenarios ξ^i $(i = 1, \ldots, N)$, we first estimate the mean vector μ and the covariance matrix Σ using monthly return data of stocks in Standard & Pool's 500 index between October 2000 and October 2010.
- We then generate N scenario vectors ξ^i $(i = 1, \ldots, N)$ from the normal distribution $N(\mu, \Sigma)$.
- In each problem instance, the prescribed return level R is set equal to 7% and the confidence level $1 - \epsilon$ is set equal to $0.85, 0.90, 0.95$, respectively.

Comparison results for probabilistically constrained problems

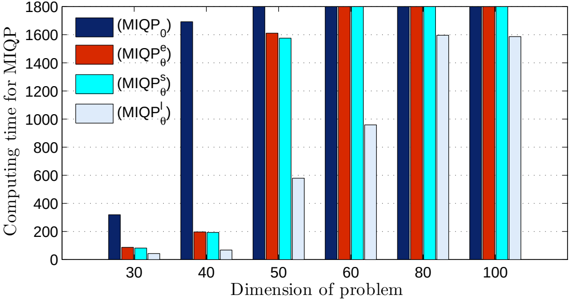

Figure: Average computing time for MIQP reformulations ($\epsilon = 0.10$)

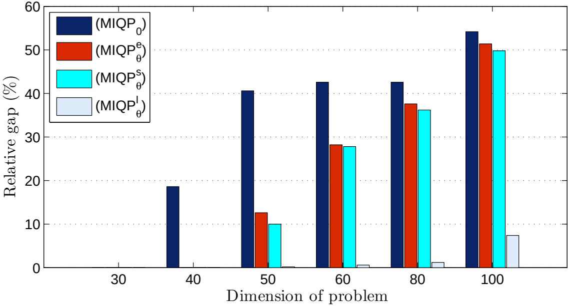

Figure: Average relative gap (%) achieved by CPLEX 12.1($\epsilon = 0.10$)

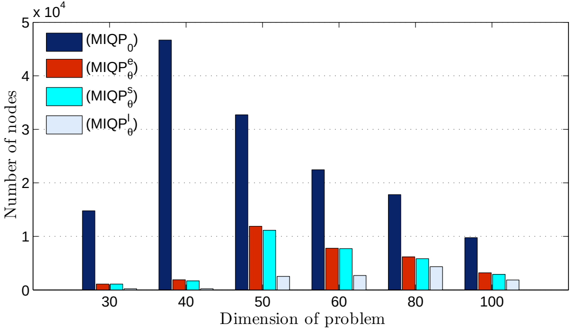

Figure: Average number of nodes explored by `CPLEX 12.1` ($\epsilon = 0.10$)

Conclusions

- Tighter SOCP and SDP relaxations for probabilistically constrained QP can be constructed using Lagrangian decomposition and lifting techniques from an equivalent MIQP reformulation.
- This MIQP reformulation is more efficient than the standard MIQP reformulation in the sense that the continuous relaxation of the reformulation is tighter or as least as tight as that of the standard MIQP reformulation.
- Computational difficulty arises when the number of scenarios (N) is large which leads to a large-size (number of constraints) MIQP.
- One of the open questions for the MIQP reformulation of probabilistically constrained QP is how to reduce the number of scenario constraints in MIQP using polyhedral properties of the constraints: surrogate scendarios?, scenario clustering? ...